COMPUTATION OF COHOMOLOGY OF LIE SUPERALGEBRAS OF VECTOR FIELDS

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The cohomology of Lie (super)algebras has many important applications in mathematics and physics. It carries most fundamental ("topological") information about algebra under consideration. At present, because of the need for very tedious algebraic computation, the explicitly computed cohomology for different classes of Lie (super)algebras is known only in a few cases. That is why application of computer algebra methods is important for this problem. We describe here an algorithm and its C implementation for computing the cohomology of Lie algebras and superalgebras. The program can proceed finite-dimensional algebras and infinite-dimensional graded algebras with finite-dimensional homogeneous components. Among the last algebras Lie algebras and superalgebras of formal vector fields are most important. We present some results of computation of cohomology for Lie superalgebras of Buttin vector fields and related algebras. These algebras being super-analogs of Poisson and Hamiltonian algebras have found many applications to modern supersymmetric models of theoretical and mathematical physics.

Keywords: Cohomology, Lie superalgebras, Antibracket, Buttin superalgebra, Symbolic computation.

1. Introduction and basic definitions

There are many applications of the Lie (super)algebra cohomology in mathematics: characteristic classes of foliations; invariant differential operators; MacDonald-type combinatorial identities, etc. (see[1] for details). The use of cohomology in theoretical and mathematical physics can be illustrated by the following applications:

- construction of the central extensions and deformations for Lie superalgebras;
- construction of the supergravity equations for N-extended Minkowski superspaces and search for possible models for these superspaces;
- study of stability for nonholonomic systems like ballbearings, gyroscopes, electro-mechanical devices, waves in plasma, etc.;
- description of an analogue of the curvature tensor for nonlinear nonholonomic constraints;[2]
- new methods for the study of integrability of dynamical systems;

- construction of so-called *higher order Lie algebras*[3] which allow in turn to construct the *Nambu mechanics*[4] generalizing the ordinary Hamiltonian mechanics;
- construction of possible invariant effective actions of Wess-Zumino-Witten type and the study of anomalies.[5]

General definitions and properties of cohomology of Lie algebras and superalgebras are described in book.[1] Let us recall briefly some basic definitions.

A *Lie superalgebra* is a $\mathbb{Z}_2$-graded algebra over a commutative ring K with a unit:

$$L = L_{\bar{0}} \oplus L_{\bar{1}},\ u \in L_\alpha,\ v \in L_\beta,\ \alpha, \beta \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\} \Longrightarrow [u, v] \in L_{\alpha+\beta}$$

The elements of $L_{\bar{0}}$ and $L_{\bar{1}}$ are called *even* and *odd*, respectively. By definition, the *Lie product* (shortly, *bracket*) $[\cdot,\cdot]$ satisfies the following axioms

$$\begin{aligned} [u, v] &= -(-1)^{p(u)p(v)}[v, u], && skew-symmetry, \\ [u, [v, w]] &= [[u, v], w] + (-1)^{p(u)p(v)}[v, [u, w]], && \text{Jacobi identity}, \end{aligned}$$

where $p(a)$ is the parity of element $a \in L_{p(a)}$. We shall assume that K is a field. If K is a field of characteristic 2 or 3, extra axioms are needed: $[u, u] = 0$ for even u in characteristic 2 and $[v, [v, v]] = 0$ for odd v in characteristic 3. To provide connection with enveloping algebra, characteristic 2 requires also the existence of a *quadratic operator* q mapping odd elements of the algebra into even ones such that

$$\begin{aligned} &q(\alpha u) = \alpha^2 q(u), \\ &[u, v] = q(u + v) - q(u) - q(v), \\ &[u, [u, v]] = [q(u), v], \end{aligned}$$

where $\alpha \in K$ and u, v are odd.

A *module* over a Lie superalgebra A is a vector space M (over the same field K) with a mapping $A \times M \to M$, such that $[a_1, a_2]m = a_1(a_2 m) - (-1)^{p(a_1)p(a_2)} a_2(a_1 m)$, where $a_1, a_2 \in A$, $m \in M$. The most important (and easy for program implementation) are *trivial* (M is arbitrary vector space, e.g., $M = K$; $am = 0$), *adjoint* ($M = A; am = [a, m]$) and *coadjoint* ($M = A'; am = \{a, m\}$ is coadjoint action) modules.

A *cochain complex* is a sequence of linear spaces C^k with linear mappings d^k

$$0 \to C^0 \xrightarrow{d^0} \cdots \xrightarrow{d^{k-2}} C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \xrightarrow{d^{k+1}} \cdots, \tag{1}$$

where the linear space $C^k = C^k(A; M)$ is a super skew-symmetric k-linear mapping $A \times \cdots \times A \to M$, $C^0 = M$ by definition. The super skew-symmetry means symmetry w.r.t. transpositions of odd adjacent elements of A and antisymmetry for all other transpositions of adjacent elements. Elements of C^k are called *cochains*.

The linear mapping d^k (or, briefly, d) is called a *differential* and satisfies the following property: $d^k \circ d^{k-1} = 0$ (or $d^2 = 0$).

The cochains mapped into zero by the differential are called *cocycles*, i.e., the space of cocycles is

$$Z^k = \mathrm{Ker}\ d^k = \{C^k \mid dC^k = 0\}.$$

The cochains which can be represented as differentials of other cochains are called *coboundaries*, i.e., the space of coboundaries is

$$B^k = \mathrm{Im}\ d^{k-1} = \{C^k \mid C^k = dC^{k-1}\}.$$

Any coboundary is, obviously, a cocycle.

The non-trivial cocycles, i.e., those which are not coboundaries, form the *cohomology*. In other words, the cohomology is the quotient space

$$H^k(A; M) = Z^k/B^k.$$

As is seen from the initial part of cochain complex (1) the basis elements of module M can be considered as non-trivial 0-cocycles.

The explicit form of the differential for a Lie superalgebra is

$$\begin{aligned}
& dC(e_0, \dots, e_q; O_{q+1}, \dots, O_k) \ = \\
& \sum_{i<j}^{q} (-1)^j C(e_0, \dots, e_{i-1}, [e_i, e_j], \dots, \widehat{e_j}, \dots, e_q; O_{q+1}, \dots, O_k) \ + \\
& (-1)^{q+1} \sum_{i=0}^{q} \sum_{j=q+1}^{k} C(e_0, \dots, e_{i-1}, [e_i, O_j], \dots, e_q; O_{q+1}, \dots, \widehat{O_j}, \dots, O_k) \ + \\
& (-1)^{i+1} \sum_{i=q+1}^{k-1} \sum_{j=q+2}^{k} C(e_0, \dots, e_q; O_{q+1}, \dots, O_{i-1}, [O_i, O_j], \dots, \widehat{O_j}, \dots, O_k) \\
& \quad + \sum_{i=0}^{q} (-1)^{i+1} e_i C(e_0, \dots, \widehat{e_i}, \dots, e_q; O_{q+1}, \dots, O_k) \\
& \quad + (-1)^q \sum_{i=q+1}^{k} O_i C(e_0, \dots, e_q; O_{q+1}, \dots, \widehat{O_i}, \dots, O_k).
\end{aligned}$$

Here e_i and O_i are even and odd elements of the algebra, respectively, and the hat "$\widehat{\ }$" marks the omitted elements.

Here are some properties and statements we use in the sequel.

An algebra and a module are called *graded* if they can be presented as sums of homogeneous components in a way compatible with the algebra bracket and the action of the algebra on the module:

$$A = \oplus_{g \in G} A_g,\ M = \oplus_{g \in G} M_g,\ [A_{g_1}, A_{g_2}] \subset A_{g_1+g_2},\ A_{g_1} M_{g_2} \subset M_{g_1+g_2},$$

where G is some abelian (semi)group. We assume $G = \mathbb{Z}$ in this paper. To avoid confusion, we use in the sequel the terms *grade* and *degree* for element of G and

number of cochain arguments, respectively. The grading in the algebra and module induces a grading on cochains and, hence, in the cohomology:

$$C^*(A;M) = \oplus_{g\in G}\, C^*_g(A;M), \quad H^*(A;M) = \oplus_{g\in G}\, H^*_g(A;M).$$

This property allows one to compute the cohomology separately for different homogeneous components; this is especially useful when the homogeneous components are finite-dimensional.

If there is an element $a_0 \in A$, such that eigenvectors (with the same eigenvalues for a given grade) of the operator $a \mapsto [a_0, a]$ form a (topological) basis of algebra A, then $H^*(A) \simeq H^*_0(A)$. In other words, all the non-trivial cocycles of the cohomology in the trivial module lie in the zero grade component. The element a_0 is called an *internal grading element*. If eigenvectors of the operator $m \mapsto a_0 m$ form also a topological basis of module M, then the same statement holds for the cohomology in the module M: $H^*(A;M) \simeq H^*_0(A;M)$.

In the case of trivial module, the exterior multiplication of cochains provides the cohomology with a structure of graded ring, i.e., if C^k and C^m are cocycles then $C^{k+m} = C^k \wedge C^m$ is also a cocycle.

If algebra A contains a *central element* Z, i.e., $[Z, a] = 0$ for any $a \in A$, then cochain $C^{k+1}(a_1, \ldots, a_k, Z) = C^k(a_1, \ldots, a_k) \wedge C^1(Z)$ is a cocycle provided that C^k is a cocycle: $dC^k = 0 \Rightarrow dC^{k+1} = 0$ because $dC(a, Z) \sim C([Z, a]) = 0$ and the differential d acts on a product of cochains as a (super)differentiation. Due to this fact any cocycle $C^k(a_1, \ldots, a_k)$ for algebras with *odd* center leads to an infinite set of cocycles of the form $C^{k+m}(a_1, \ldots, a_k, \underbrace{Z, \ldots, Z}_{m})$. We shall encounter such situation later in our computations.

There are also another multiplicative structures in the cohomology theory, but we shall not use them in this work.

2. Lie superalgebras of vector fields

Below a list of the main Lie superalgebras of formal vector fields is given.[6] We consider some sets of even (x_i, q_i, p_i, t) and odd (called also *Grassmann*) variables (X_i, T). In many cases the vector fields can be expressed in terms of generating functions. The coordinates of vector fields and generating functions are assumed to be formal power series in the even and odd variables. Note that all the algebras depending only on the odd variables are finite-dimensional. All these algebras are graded due to a prescribed grading of the variables. There are some *standard* gradings for the variables: all variables x_i, q_i, p_i, X_i have grade 1 and the separate variables t, T have grade 2. *Non-standard* gradings with zero or negative grades for some odd variables are possible (and useful, as we show below) too. The divergence-free algebras are called *special*. The symbol $\mathbf{Z}$ denotes the 1-dimensional center of an algebra consisting of constants in terms of generating function, i. e., $\mathbf{Z} = \mathrm{Span}(\{1\})$.

1. *General vectorial superalgebra* $\mathbf{W(n \mid m)}$ or $\mathbf{vect(n \mid m)}$
   Variables: $x_1, \ldots, x_n; X_1, \ldots, X_m$
   The bracket denotes the supercommutator of vector fields of the form
   $\sum\limits_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum\limits_{k=1}^{m} g_k \frac{\partial}{\partial X_k}$

2. *Special vectorial superalgebra* $\mathbf{S(n \mid m)}$ or $\mathbf{svect(n \mid m)}$ consists of the elements from $\mathbf{W(n \mid m)}$ satisfying the divergence-free condition
   $\sum\limits_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum\limits_{k=1}^{m} (-1)^{p(g_k)} \frac{\partial g_k}{\partial X_k} = 0$

3. *Poisson superalgebra* $\mathbf{Po(2n \mid m)}$
   Variables: $p_1, \ldots, p_n, q_1, \ldots, q_n; X_1, \ldots, X_m$
   Bracket: $\{f, g\}_{Pb} = \sum\limits_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) - (-1)^{p(f)} \sum\limits_{k=1}^{m} \frac{\partial f}{\partial X_k} \frac{\partial g}{\partial X_k}$
   Sometimes it is more convenient to redenote the odd variables X_k and set $P_k = \frac{1}{2}(X_k - X_{r+k}), Q_k = \frac{1}{2}(X_k + X_{r+k})$ for $k \leq r = [m/2]$,
   ($U = X_{2r+1}$ for m odd) and the last sum in the bracket takes the form $\sum\limits_{k=1}^{r} (\frac{\partial f}{\partial P_k} \frac{\partial g}{\partial Q_k} + \frac{\partial f}{\partial Q_k} \frac{\partial g}{\partial P_k})$ (one should add the term $\frac{\partial f}{\partial U} \frac{\partial g}{\partial U}$ for m odd).
   Hamiltonian superalgebra is $\mathbf{H(2n \mid m) = Po(2n \mid m)/Z}$
   Special Hamiltonian superalgebra $\mathbf{SH(0 \mid m)}$ is a simple ideal of codimension one in $\mathbf{H(0 \mid m)}$

4. *Contact superalgebra* $\mathbf{K(2n+1 \mid m)}$
   Variables: $t, p_1, \ldots, p_n, q_1, \ldots, q_n; X_1, \ldots, X_m$
   Bracket: $\{f, g\}_{Kb} = \delta(f) \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t} \delta(g) - \{f, g\}_{Pb}$
   $\delta(f) = 2f - E(f),\ E = \sum\limits_{i=1}^{n} \left(p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i} \right) + \sum\limits_{k=1}^{m} X_k \frac{\partial}{\partial X_k}$

5. *Buttin superalgebra* $\mathbf{B(n)}$
   Variables: $x_1, \ldots, x_n; X_1, \ldots, X_n$
   Bracket:
   $$\{f, g\}_{Bb} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial X_i} + (-1)^{p(f)} \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial x_i} \right) \tag{2}$$
   Leites superalgebra is $\mathbf{Le(n) = B(n)/Z}$

6. *Special Buttin superalgebra* $\mathbf{SB(n)}$ is subalgebra of $\mathbf{B(n)}$
   subject to the constraint $\Delta f = 0$ for generating function, where
   $$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial X_i}. \tag{3}$$
   Special Leites superalgebra is $\mathbf{SLe(n) = SB(n)/Z}$

7. *Odd contact superalgebra* $\mathbf{M(n)}$
   Variables: $x_1, \dots, x_n; T, X_1, \dots, X_n$
   Bracket: $\{f,g\}_{Mb} = \delta(f)\frac{\partial g}{\partial T} + (-1)^{p(f)}\frac{\partial f}{\partial T}\delta(g) - \{f,g\}_{Bb}$
   $\delta(f) = 2f - E(f),\ E = \sum\limits_{i=1}^{n}\left(x_i\frac{\partial}{\partial x_i} + X_i\frac{\partial}{\partial X_i}\right)$
8. *Special odd contact superalgebra* $\mathbf{SM(n)}$ is subalgebra of $\mathbf{M(n)}$ subject to the constraint $(1-E)\frac{\partial f}{\partial T} - \Delta f = 0$ for generating function.

3. Outline of algorithm and its implementation

To compute the cohomology one needs to solve the equation

$$dC^k = 0, \tag{4}$$

and throw away those solutions of (4) which can be expressed in the form

$$C^k = dC^{k-1}.$$

In the case of finite-dimensional Lie superalgebras determining equation (4) is a system of finite-dimensional homogeneous linear algebraic equations. In the case of infinite-dimensional graded Lie superalgebras, such as Lie superalgebras of vector fields with even variables, equation (4) is a system of linear homogeneous functional equations with integer arguments. Unfortunately there is no general method for solving such systems in closed form, though in a few exceptional cases such solutions are known.

If the grading leads to finite-dimensional space of cochains in a given grade, one can proceed just as in the case of finite-dimensional algebra. Unfortunately the very important case of computation of cohomology in adjoint module for infinite-dimensional algebras can not be reduced to the set of finite-dimensional tasks at any choice of grading: in the case of adjoint module the cochains contain both elements of algebra and dual elements, these elements inevitably should have opposite grades. Nevertheless, there are important problems (such as the Spencer cohomology playing an essential role in the formal theory of differential equations[7]) requiring computation of cohomology in adjoint module with respect to finite-dimensional subalgebras of infinite-dimensional algebras.

There are several packages for computing cohomology of Lie algebras and superalgebras written in *Reduce* [8,9] and *Mathematica*.[2] Some new results were obtained completely or partially with the help of these packages.[a] However, abilities of these packages are restricted by rather small problems. We wrote a more advanced program[11] which allows us to consider more difficult and real problems.

[a]In particular, D. Leites informed us that A. Shapovalov discovered one of the cocycles from the cohomology of special Hamiltonian superalgebra SH(0|4) with the help of the *Mathematica* program[2] written by P. Grozman. This cocycle was missed by D. Leites and D. Fuchs when they investigated this cohomology[10] by hand.

The C code of the program, of total length near 14200 lines, contains about 400 functions realizing top level algorithms, simplification of indexed objects, working with Grassmannian objects, exterior calculus, linear algebra, substitutions, list processing, input and output, etc.

All operations with scalar coefficients, including input and output, are localized in 17 functions which formats do not depend on the nature of field of scalars K. Reading from the input which field should be used, the program assigns the suitable function addresses to the corresponding function pointer variables. This feature of the C language allows to carry out computations over arbitrary fields without recompiling and any loss of efficiency. Up to now we have implemented rational numbers of arbitrary precision, i.e., the field $\mathbb{Q}$, its complex extension $\mathbb{Q}[i]$,[b] rational functions of arbitrary parameters (for classification problems) and the fields $\mathbb{Z}_p$.[c] Of course, other fields can be easily added if necessary.

We represent Grassmann monomials by integer numbers using one-to-one correspondence between (binary codes of) non-negative integers and Grassmann monomials. This representation allows one efficiently to implement the operations with Grassmann monomials by means of the basic computer commands.

The program performs sequentially the following steps:

1. *Reading input information.*

2. *Constructing a basis* for the algebra. The basis can be read from the input file; otherwise the program constructs it from the definition of the algebra. Non-trivial computations at this step arise only in the case of divergence-free algebras. The basis elements of such algebras should satisfy some conditions. In fact, we should construct the basis elements of the subspace given by a system of linear equations. The task is thereby reduced to some problem of linear algebra combined with shifts of indices. For example, among the divergence-free conditions for the special Buttin algebra SB(3) there are the following two equations

$$ia_{ijk;XY} - (k+1)a_{i-1,j,k+1;YZ} = 0,$$

$$ia_{ijk;XZ} + (j+1)a_{i-1,j+1,k;YZ} = 0.$$

   Here $a_{ijk;XY}, \ldots$ are coefficients at the monomials $x^i y^j z^k XY, \ldots$ in the generating function; x, y, z and X, Y, Z are even and odd variables, respectively. First of all, we have to shift indices j and k in the second equation to reduce the last terms of both equations to the same multiindices. Then, using some

[b] Note that the fields $\mathbb{R}$ and $\mathbb{C}$ being non-constructive objects do not admit a computer implementation at all.

[c] For efficiency reasons the prime p should not exceed 46337 on 32bit and 3037000493 on 64bit computers.

simple tricks of linear algebra, we can easily construct the corresponding basis element

$$O^1_{ijk} = kx^iy^jz^{k-1}XY - jx^iy^{j-1}z^kXZ + ix^{i-1}y^jz^kYZ.$$

As a result for SB(3) we have the basis:

$$\begin{array}{lcl}
(1)\ E^1 & = & XYZ \\
(2)\ E^2_{ijk} & = & kx^iy^jz^{k-1}X - ix^{i-1}y^jz^kZ \\
(3)\ E^3_{ijk} & = & kx^iy^jz^{k-1}Y - jx^iy^{j-1}z^kZ \\
(4)\ O^1_{ijk} & = & kx^iy^jz^{k-1}XY - jx^iy^{j-1}z^kXZ + ix^{i-1}y^jz^kYZ \\
(5)\ O^2_{ijk} & = & x^iy^jz^k
\end{array}$$

3. *Constructing the commutator table* for the algebra (if this table has not been read from the input file).
   We illustrate this step by non-zero commutators of SB(3) generated by the program:

$$\begin{array}{llcl}
(1) & [E^2_{ijk}, E^2_{lmn}] & = & (ni - lk)E^2_{i+l-1,j+m,k+n-1} \\
(2) & [E^2_{ijk}, E^3_{lmn}] & = & \frac{nkj-mk^2+mk}{n+k-1}E^2_{i+l,j+m-1,k+n-1} \\
 & & & +\frac{n^2i-nlk-ni}{n+k-1}E^3_{i+l-1,j+m,k+n-1} \\
(3) & [E^3_{ijk}, E^3_{lmn}] & = & (nj - mk)E^3_{i+l,j+m-1,k+n-1} \\
(4) & [E^2_{ijk}, O^1_{lmn}] & = & (ni - lk)O^1_{i+l-1,j+m,k+n-1} \\
(5) & [E^3_{ijk}, O^1_{lmn}] & = & (nj - mk)O^1_{i+l,j+m-1,k+n-1} \\
(6) & [E^1, O^2_{ijk}] & = & -O^1_{ijk} \\
(7) & [E^2_{ijk}, O^2_{lmn}] & = & (ni - lk)O^2_{i+l-1,j+m,k+n-1} \\
(8) & [E^3_{ijk}, O^2_{lmn}] & = & (nj - mk)O^2_{i+l,j+m-1,k+n-1} \\
(9) & [O^1_{ijk}, O^2_{lmn}] & = & \frac{nj-mk}{n+k}E^2_{i+l,j+m-1,k+n} + \frac{-ni+lk}{n+k}E^3_{i+l-1,j+m,k+n}
\end{array}$$

4. *Creating the general form* of expressions for coboundaries and determining equations for cocycles.

5. *Transition to a particular grade* in general expressions. At this step expressions for coboundaries take the form $\mathbf{x} = \mathbf{bt}$, equations for cocycles take the form $\mathbf{Zx} = \mathbf{0}$, where vector $\mathbf{x}$ corresponds to C^k, parameter vector $\mathbf{t}$ corresponds to C^{k-1}, matrices $\mathbf{Z}, \mathbf{b}$ correspond to the differential d. All these vector spaces are finite-dimensional for any particular grade.

6. *Computing the quotient space* $H^k(A; M) = Z^k/B^k$. Here the cocycle subspace Z^k is given by relations $\mathbf{Zx} = \mathbf{0}$, and the coboundary subspace B^k is given parametrically by $\mathbf{x} = \mathbf{bt}$.

   Substeps:

   (a) Eliminate $\mathbf{t}$ from $\mathbf{x} = \mathbf{bt}$ to get equations $\mathbf{Bx} = \mathbf{0}$

(b) Reduce both relations $\mathbf{Bx} = \mathbf{0}$ and $\mathbf{Zx} = \mathbf{0}$ to the canonical (row echelon) form by Gauss elimination. If rank$\mathbf{B}$ = rank$\mathbf{Z}$, then there is no non-trivial cocycle; otherwise go to Substep (6c).

(c) Set $\mathbf{Bx} = \mathbf{y}$ and substitute these relations into $\mathbf{Zx} = \mathbf{0}$ to get relations $\mathbf{Ay} = \mathbf{0}$. The *parametric* (non-leading) y's of the last relations are non-trivial cocycles; that is, they form a basis of the cohomology.

In fact, the above procedure is based on the relation for quotient spaces

$$Z/B = \frac{Y/B}{Y/Z},$$

where Y is an artificially introduced space, combining the above x's and y's.

7. *Output the non-trivial cocycles.* The program can output results in 2D ASCII, LaTeX and standard for usual computer algebra systems 1D forms. The last form of output is useful for investigating the structure of cohomology ring with the help of the systems like *Maple*, *Mathematica* or *Reduce*. The operations in such investigations (multiplications and comparisons of cochains) are not difficult from computational point of view, but interactive abilities of the above systems are very convenient for the analysis of the cohomology ring.

To split the whole task to smaller ones Steps (4–7) are executed separately for even and odd parts of the cochain complex.

3.1. *Example of output file: Computation of* $H_0^5(\mathrm{SLe}(2))$

The below output demonstrates computation of 5-cocycle in grade 0 from cohomology in trivial module for special Leites superalgebra of vector fields in superdimension $(2|2)$. Here are some explanations to this output. The brackets $< \cdots >$ include a comment in input file. Some elements of input are optional. If they are omitted, the program either constructs some standard ones or asks to input them from keyboard. $g(a)$ is a $\mathbb{Z}$-grade of element a. E_{ij} and O^1, O^2_{ij} are even and odd basis elements of superalgebra. $'E_{ij}$, $'O^1$, $'O^2_{ij}$ are dual elements to corresponding basis elements. The vertical and horizontal dots mean that we have omitted for brevity some long (sub)expressions in this illustrative example. Output of determining equations for cocycles may be useful. Sometimes (for a low degree cohomology) one can see the general solution for these equations. Almost all elements of output (excepting the resulting cocycle) can be suppressed by corresponding settings in the initiating file. t_i are arbitrary parameters describing the space of coboundaries. This computation gives one basis element of the cohomology, but the program produces also its four equivalent forms: one can choose any of them (or their linear combination) in order to get more compact or symmetrical expression. Note that in the case of several basis elements of cohomology, these alternative forms may be linear combinations of the original ones.

```
Input file: D:\Kornyak\LieCohomology\In\test.in
Input data:
<* Special Leites superalgebra SLe(n) = SB(n)/Z *>
Even variables: x; y. < Optional >
Grading for even variables: 1; 1. < Optional >
Odd variables:  X; Y. < Optional >
Grading for odd variables: -1; -1. < Optional >
Module type: Trivial. < Coadjoint Adjoint>
Special Leites superalgebra: 2.
Cohomology number: 5. < Optional >
Grade: 0. < Optional >
```

Even variables of vector field: $x\ y$; $g(x) = 1\ g(y) = 1$.
Odd variables of vector field: $X\ Y$; $g(X) = -1\ g(Y) = -1$.
Basis elements of Lie superalgebra:

$$\begin{array}{llll} (1) & E_{ij} & = & jx^iy^{j-1}X - ix^{i-1}y^jY; \quad g(E_{ij}) = i+j-2; \quad i \geq 0,\ j \geq 0. \\ (2) & O^1 & = & XY; \quad g(O^1) = -2 \\ (3) & O^2_{ij} & = & x^iy^j; \quad g(O^2_{ij}) = i+j; \quad i \geq 0,\ j \geq 0. \end{array}$$

Non-zero commutators of Lie superalgebra:

$$\begin{array}{llll} (1) & [E_{ij}, E_{kl}] & = & (li-kj)E_{i+k-1,j+l-1} \\ (2) & [E_{ij}, O^2_{kl}] & = & (li-kj)O^2_{i+k-1,j+l-1} \\ (3) & [O^1, O^2_{ij}] & = & -E_{ij} \end{array}$$

Expression for even coboundaries:

$$\begin{aligned} dC^4_{\bar 0} = & \{(ro-qp)C\left(E_{ij}, E_{kl}, E_{mn}, E_{o+q-1,p+r-1}\right) + \cdots \\ & \cdots + (-ri+qj)C\left(O^2_{i+q-1,j+r-1}, O^2_{kl}, O^2_{mn}, O^2_{op}\right)\} \\ & {}'E_{ij} \wedge' O^2_{kl} \wedge' O^2_{mn} \wedge' O^2_{op} \wedge' O^2_{qr} \end{aligned}$$

Expression for odd coboundaries:

$$\begin{aligned} dC^4_{\bar 1} = & \left\{(-pm+on)C\left(E_{ij}, E_{kl}, E_{m+o-1,n+p-1}, O^1\right) + \cdots \right. \\ & \left. \cdots + C\left(E_{op}, O^2_{ij}, O^2_{kl}, O^2_{mn}\right)\right\}' O^1 \wedge' O^2_{ij} \wedge' O^2_{kl} \wedge' O^2_{mn} \wedge' O^2_{op} \end{aligned}$$

Determining equation for even cocycles:

$$\begin{aligned} dC^5_{\bar 0} = & \{(-tq+sr)C\left(E_{ij}, E_{kl}, E_{mn}, E_{op}, E_{q+s-1,r+t-1}\right) + \cdots \\ & \cdots + C\left(E_{qr}, O^2_{ij}, O^2_{kl}, O^2_{mn}, O^2_{op}\right)\}' O^1 \wedge' O^2_{ij} \wedge' O^2_{kl} \wedge' O^2_{mn} \wedge' O^2_{op} \wedge' O^2_{qr} \end{aligned}$$

Determining equation for odd cocycles:

$$\begin{aligned} dC^5_{\bar 1} = & \left\{(ro-qp)C\left(E_{ij}, E_{kl}, E_{mn}, E_{o+q-1,p+r-1}, O^1\right) + \cdots \right. \\ & \left. \cdots + (-ti+sj)C\left(O^2_{i+s-1,j+t-1}, O^2_{kl}, O^2_{mn}, O^2_{op}, O^2_{qr}\right)\right\} \\ & {}'E_{ij} \wedge' O^2_{kl} \wedge' O^2_{mn} \wedge' O^2_{op} \wedge' O^2_{qr} \wedge' O^2_{st} \end{aligned}$$

Coboundary component expressions in grade 0:

$$\begin{array}{rcl} C\left(E_{01}, E_{02}, E_{03}, E_{10}, E_{12}\right) & = & 2t_1 - 3t_3 \\ \vdots & \vdots & \vdots \\ C\left(E_{10}, O^1, O^2_{10}, O^2_{10}, O^2_{10}\right) & = & 0 \end{array}$$

where

$$\begin{array}{rcl} t_1 & = & C\left(E_{01}, E_{02}, E_{03}, E_{11}\right) \\ \vdots & \vdots & \vdots \\ t_{246} & = & C\left(O^1, O^1, O^2_{20}, O^2_{20}\right) \end{array}$$

Even cocycles in grade 0 are trivial.
Coboundary component expressions in grade 0:

$$\begin{array}{rcl} C\left(E_{01}, E_{02}, E_{03}, E_{04}, O^1\right) & = & 0 \\ \vdots & \vdots & \vdots \\ C\left(O^1, O^1, O^2_{10}, O^2_{10}, O^2_{20}\right) & = & 4t_{239} + 2t_{245} \end{array}$$

where

$$\begin{array}{rcl} t_1 & = & C\left(E_{01}, E_{02}, E_{05}, O^1\right) \\ \vdots & \vdots & \vdots \\ t_{245} & = & C\left(E_{20}, O^1, O^2_{10}, O^2_{10}\right) \end{array}$$

Odd cocycles in grade 0:

$$\begin{array}{rcl} (1)\ a^5_0 & = & C\left(E_{02}, E_{10}, E_{11}, E_{20}, O^2_{01}\right) - 2C\left(E_{10}, E_{11}, O^1, O^2_{02}, O^2_{10}\right) \\ & & -C\left(E_{11}, E_{20}, O^1, O^2_{01}, O^2_{01}\right) \\ & = & C\left(2yX, -Y, xX - yY, -2xY, y\right) - 2C\left(-Y, xX - yY, XY, y^2, x\right) \\ & & -C\left(xX - yY, -2xY, XY, y, y\right) \end{array}$$

and also:

$$\begin{array}{ll} (1) & C\left(E_{01}, E_{02}, E_{10}, E_{11}, O^2_{20}\right) + 4C\left(E_{10}, E_{11}, O^1, O^2_{01}, O^2_{11}\right) \\ & -2C\left(E_{10}, E_{11}, O^1, O^2_{02}, O^2_{10}\right) \\ & = C\left(X, 2yX, -Y, xX - yY, x^2\right) + 4C\left(-Y, xX - yY, XY, y, xy\right) \\ & -2C\left(-Y, xX - yY, XY, y^2, x\right) = a^5_0 \\ \vdots & \\ (4) & C\left(E_{01}, E_{10}, E_{11}, E_{20}, O^2_{02}\right) - 4C\left(E_{10}, E_{11}, O^1, O^2_{01}, O^2_{11}\right) \\ & -C\left(E_{11}, E_{20}, O^1, O^2_{01}, O^2_{01}\right) \\ & = C\left(X, -Y, xX - yY, -2xY, y^2\right) - 4C\left(-Y, xX - yY, XY, y, xy\right) \\ & -C\left(xX - yY, -2xY, XY, y, y\right) = a^5_0 \end{array}$$

4. Buttin vector fields and related algebras

The Poisson and Hamiltonian (super)algebras are very important algebras of vector fields. In the papers on the *deformation quantization*[12,13] it was proven that the Poisson bracket is an *unique* structure providing deformation of a commutative algebra of differentiable functions on a manifold to a new noncommutative but associative algebra. In paper[14] we present some results about the structure of cohomology rings for Poisson, Hamiltonian and related algebras obtained with the help of our program.

Whereas the Poisson bracket, defined on a $2n$-dimensional symplectic manifold, has an old history, its counterpart called Buttin bracket (or odd Poisson bracket, or antibracket) and defined on a $(n|n)$-dimensional *odd symplectic* supermanifold[d] is a comparatively new construction. The first example of such bracket has appeared in Schouten's paper[17] as an extension of the Lie bracket on vector fields to an bracket on skew-symmetric contravariant (i.e., tangent) tensor fields (*multivectors*). A more abstract formulation for this bracket was given by Buttin.[18] Since 1981 antibrackets are very popular in theoretical physics, because they play the crucial role in the Batalin-Vilkovisky (BV) covariant method for quantizing general gauge theories.[19] This method called the BV (or antibracket, or field-antifield) formalism[e] being currently a most powerful procedure for quantizing gauge theories is applied also in string and topological field theories.[20]

Therefore investigation of properties of Buttin and related algebras is a problem of interest for physics. We should stress also importance of the special subalgebras SB($*$) and SLe($*$), because Δ-operator (3) plays an essential role in the BV formalism: so-called *master equation* in the BV formalism is defined via this operator.

4.1. *Computations*

Here we present some results of computations of cohomologies in the trivial module for the algebras Buttin B(1), special Buttin SB(1) and their centerless quotients Le(1) and SLe(1). We present also the results for odd contact algebra M(1) and its special subalgebra SM(1). Our computations are restricted with values for cohomology degree (≤ 10) and grade (≤ 10). We computed also some cocycles for the case $n \geq 1$. Here we mention only the most regular of them: $a^1_{-n} = C(X_1 \cdots X_n)$ for the algebras SB(n) and SLe(n), this cocycle generates an infinite number of its wedged powers if n is even, and $a^2_0 = C(X_1, x_1) = \ldots = C(X_n, x_n)$ for the algebras Le(n) and SLe(n). Any cocycle for the algebra SB(n) generates also new cocycles due to presence of odd center in this algebra.

We use the following grading for the variables: $g(x_i) = 1,\ g(X_i) = -1,\ g(T) = 0.$

---

[d]Such manifolds possess interesting and unusual geometrical properties.[15,16]

[e]The antibracket in BV formalism is an odd symplectic form on the (infinite-dimensional) space of fields and antifields playing the role of even and odd variables respectively, and the partial derivatives in formulas (2) and (3) should be replaced by variational derivatives and the summation by integration.

With this grading the algebras B(n), Le(n) and M(n) contain an internal grading element at any n, i. e., there is no need to compute cohomology in grades different from zero for this algebras. Unfortunately, there is no good grading providing internal grading element for the special subalgebras.[f] One can see also that the even generating functions correspond to the odd elements of algebra and vice versa. In particular, the element 1 is an *odd* central element in the algebras B(n) and SB(n).

In the below formulas all indices $i, j \geq 0$, but for the algebras Le(1) and SLe(1) the central element $O_0 = 1$ should be excluded.

4.1.1. $H^k(\mathrm{B}(1))$ *and* $H^k(\mathrm{Le}(1))$

Basis elements:

$$\begin{array}{llcll} (1) & E_i & = & x^i X; & g(E_i) = i+1; \\ (2) & O_i & = & x^i; & g(O_i) = i. \end{array}$$

Non-zero commutators:

$$\begin{array}{llcl} (1) & [E_i, E_j] & = & (i-j)E_{i+j-1} \\ (2) & [E_i, O_j] & = & -jO_{i+j-1} \end{array}$$

Generating cocycles for $H^k(\mathrm{B}(1))$:

$$\begin{aligned} a^3 &= C(X, xX, x^2X) \\ b^3 &= C(X, xX, x) - \frac{1}{2}C(X, x^2X, 1) \end{aligned}$$

The ring $H^*(\mathrm{B}(1))$ contains also all cocycles of the form $a^3 \wedge C(1) \wedge \cdots \wedge C(1)$ and $b^3 \wedge C(1) \wedge \cdots \wedge C(1)$.

$H^k(\mathrm{Le}(1))$ contains only 3 non-trivial cocycles for $k \leq 20$: the above cocycle a^3 and the cocycles $b^3 = C(X, xX, x)$ and $a^2 = C(X, x)$. The last cocycle describes the central extension of Le(1) to B(1).

4.1.2. $H^k(\mathrm{M}(1))$

Basis elements:

$$\begin{array}{llcll} (1) & E^1_i & = & x^i X; & g(E^1_i) = i-1; \\ (2) & E^2_i & = & x^i T; & g(E^2_i) = i; \\ (3) & O^1_i & = & x^i TX; & g(O^1_i) = i-1; \\ (4) & O^2_i & = & x^i; & g(O^2_i) = i. \end{array}$$

[f] There are grading elements for the algebras SB(n), SLe(n) and SM(n) for n even at the grading: $g(x_i) = 1,\ g(X_i) = -1,\ 1 \leq i \leq \frac{n}{2};\ g(x_i) = -1,\ g(X_i) = 1,\ \frac{n}{2} < i \leq n$; but in this case the space of cochains in zero grade becomes infinite-dimensional.

Non-zero commutators:

$$
\begin{array}{llcl}
(1) & [E^1_i, E^1_j] & = & (j-i)E^1_{i+j-1}\\
(2) & [E^1_i, E^2_j] & = & (-i+1)E^1_{i+j} + jE^2_{i+j-1}\\
(3) & [E^2_i, E^2_j] & = & (j-i)E^2_{i+j}\\
(4) & [E^1_i, O^1_j] & = & (j-i)O^1_{i+j-1}\\
(5) & [E^2_i, O^1_j] & = & (j-i+1)O^1_{i+j}\\
(6) & [E^1_i, O^2_j] & = & jO^2_{i+j-1}\\
(7) & [E^2_i, O^2_j] & = & (j-2)O^2_{i+j}\\
(8) & [O^1_i, O^2_j] & = & (-j+2)E^1_{i+j} + jE^2_{i+j-1}
\end{array}
$$

We have found only one non-trivial cocycle

$$
\begin{array}{rcl}
a^3 & = & C(T,TX,x) - \frac{1}{2}C(T,xTX,1)\\
& = & C(X,xX,xT) - \frac{1}{2}C(T,xTX,1)\\
& = & -C(X,T,xT) - \frac{1}{2}C(T,xTX,1)\\
& = & C(X,xTX,x) + \frac{1}{2}C(x^2X,TX,1) + \frac{1}{2}C(xT,TX,1)\\
& = & -C(xX,TX,x) - \frac{1}{2}C(T,xTX,1).
\end{array}
$$

4.1.3. $H^k_g(\mathrm{SB}(1))$ *and* $H^k_g(\mathrm{SLe}(1))$

Basis elements:

$$
\begin{array}{llcll}
(1) & E & = & X; & g(E) = -1;\\
(2) & O_i & = & x^i; & g(O_i) = i.
\end{array}
$$

Non-zero commutators:

$$
(1) \quad [E, O_i] \quad = \quad -iO_{i-1}
$$

The algebras SB(1) and SLe(1) contain the odd centers $Z = \mathrm{Span}(\{1\})$ and $Z = \mathrm{Span}(\{\mathrm{x}\})$ respectively. The sets of non-trivial cocycles consist of some generating cocycles and their consequences obtained by multiplication of these cocycles by arbitrary wedged powers of $C(1)$ and $C(x)$ for SB(1) and SLe(1) respectively.

Table 1 presents all generating non-trivial cocycles in the limits for cohomology degree $k \leq 10$ and grade $g \leq 10$ for algebras SB(1) and SLe(1). The presence of generating cocycle for SLe(1) is marked with * in the table. We give here the explicit expressions up to 3-cocycles for SB(1) (The cocycles for SLe(1) can be obtained by deleting terms with argument 1 from these expressions).

$$
\begin{array}{rcl}
a^1_{-1} & = & C(X)\\
a^3_1 & = & C(X,1,x^2) - C(X,x,x)\\
a^3_3 & = & C(X,1,x^4) - 4C(X,x,x^3) + 3C(X,x^2,x^2)\\
a^3_5 & = & C(X,1,x^6) - 6C(X,x,x^5) + 15C(X,x^2,x^4) - 10C(X,x^3,x^3)\\
a^3_7 & = & C(X,1,x^8) - 8C(X,x,x^7) + 28C(X,x^2,x^6) - 56C(X,x^3,x^5)\\
& & +35C(X,x^4,x^4)
\end{array}
$$

$$
\begin{aligned}
a_9^3 \quad = \quad & C(X,1,x^{10}) - 10C(X,x,x^9) + 45C(X,x^2,x^8) - 120C(X,x^3,x^7) \\
& +210C(X,x^4,x^6) - 126C(X,x^5,x^5)
\end{aligned}
$$

Table 1: Generating cocycles for $H_g^k(\mathrm{SB}(1))$ and $H_g^k(\mathrm{SLe}(1))$

$k \backslash g$	-1	0	1	2	3	4	5	6	7	8	9	10
1	a_{-1}^1 *											
2												
3			a_1^3 *		a_3^3 *		a_5^3 *		a_7^3 *		a_9^3 *	
4				a_2^4		a_4^4	a_5^4 *	a_6^4	a_7^4 *	a_8^4 * b_8^4	a_9^4 *	a_{10}^4 * b_{10}^4
5					a_3^5		a_5^5	a_6^5	a_7^5 * b_7^5	a_8^5	a_9^5 * b_9^5 c_9^5	a_{10}^5 * b_{10}^5
6						a_4^6		a_6^6	a_7^6	a_8^6 b_8^6	a_9^6 * b_9^6	a_{10}^6 b_{10}^6 c_{10}^6
7							a_5^7		a_7^7	a_8^7	a_9^7 b_9^7	a_{10}^7 b_{10}^7
8								a_6^8		a_8^8	a_9^8	a_{10}^8 b_{10}^8
9									a_7^9		a_9^9	a_{10}^9
10										a_8^{10}		a_{10}^{10}

4.1.4. $H_g^k(\mathrm{SM}(1))$

Basis elements:

$$
\begin{array}{llcll}
(1) & E_i & = & ix^{i-1}T + (-i+2)x^iX; & g(E_i) = i-1; \\
(2) & O^1 & = & TX; & g(O^1) = -1; \\
(3) & O_i^2 & = & x^i; & g(O_i^2) = i.
\end{array}
$$

Non-zero commutators:

$$
\begin{array}{llcl}
(1) & [E_i, E_j] & = & (2j-2i)E_{i+j-1} \\
(2) & [E_i, O_j^2] & = & (2j-2i)O_{i+j-1}^2 \\
(3) & [O^1, O_i^2] & = & E_i
\end{array}
$$

Our computations in the limits $k, g \leq 10$ show that the only non-trivial cocycles take the form $a^k_{k\text{-}2}$ excepting the case $k = 2$. We give below the explicit expressions for the first three these cocycles. The cocycle $a^1_{\text{-}1}$ generates an infinite set of cocycles $a^k_{\text{-}k} = a^1_{\text{-}1} \wedge \cdots \wedge a^1_{\text{-}1} = C(TX, \ldots, TX)$.

$$
\begin{aligned}
a^1_{\text{-}1} &= C(TX) \\
a^3_1 &= C(T + xX, 2xT, 1) - 2C(TX, x, x) \\
&= C(2X, T + xX, x^2) - 2C(TX, x, x) \\
&= -C(2X, 2xT, x) + 2C(TX, 1, x^2) + 2C(TX, x, x) \\
a^4_2 &= C(T + xX, 2xT, 1, x) - \frac{1}{6}C(T + xX, 3x^2T - x^3X, 1, 1) - \frac{4}{3}C(TX, x, x, x) \\
&= C(2X, T + xX, x, x^2) - \frac{1}{6}C(T + xX, 3x^2T - x^3X, 1, 1) - \frac{4}{3}C(TX, x, x, x) \\
&= -\frac{1}{2}C(2X, 2xT, x, x) - \frac{1}{6}C(T + xX, 3x^2T - x^3X, 1, 1) + 2C(TX, 1, x, x^2) \\
&\quad + \frac{2}{3}C(TX, x, x, x)
\end{aligned}
$$

5. Conclusion

Mathematicians regard the problem of computation of cohomology as solved if they construct the full cohomology ring. The structure of such rings may be rather complicated including many non-trivial relations between the cocycles in contrast to the examples in this paper where the rings are commutative. To get a clear idea about the structure of cohomology ring one should compute usually the cocycles up to degrees high enough. Unfortunately the computation of cohomology is a typical problem with the combinatorial explosion. Nevertheless, some results can be obtained with the help of computer having an efficient enough program. On the other hand, physicists are interested mainly in the second cohomologies describing the central extensions and deformations. Such cohomologies can be computed rather easily even for large algebras. Some essential possibilities remain for increasing the efficiency of the program and we hope to implement the corresponding improvements in future.

Acknowledgment

I would like to thank D. Leites for initiating this work and helpful communications. I am also grateful to V. Gerdt and O. Khudaverdian for fruitful discussions and useful advises. This work was supported in part by INTAS project No. 96-0842 and RFBR project No. 98-01-00101.

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